Complex analysis for EE, 2012-13, problem set 4

Note to students: Due to the current situation, some students have yet to cover the Cauchy-Goursat theorem in class. Although some different variants of this theorem exist, the main result is this: if Γ is a closed simple curve in $\mathbb{C}$, which is contained in a region Ω along with its interior (the section of $\mathbb{C}$ enclosed by it), and if $f : \Omega \to \mathbb{C}$ is holomorphic, then $\int_\Gamma f(z)dz = 0$.

The use of this theorem is required in some parts of this sheet. Problem 10 deals with this theorem directly, extending the results shown in class to an a priori different setting.

1. Evaluate the following line integrals:

   (a) $\int_\Gamma \bar{z}dz$ where Γ is the triangular curve connecting $0, 1+i, i$.
   Solution:
   We can decompose the integral into three parts:

$$\int_\Gamma \bar{z}dz = \int_{[0,1+i]} \bar{z}dz + \int_{[1+i,i]} \bar{z}dz + \int_{[i,0]} \bar{z}dz =$$
$$= \int_0^1 t(1-i)(1+i)dt + \int_0^1 (1-t-i)(-1)dt + \int_0^1 -i(1-t)(-i)dt =$$
$$= 2\left.\frac{t^2}{2}\right|_0^1 + \left(\left.\frac{t^2}{2}\right|_0^1 + (i-1)\right) + \left(\left.\frac{t^2}{2}\right|_0^1 - 1\right) = i.$$

   (b) $\int_{|z|=2} \frac{dz}{z^2-1}$.
   Solution:
   We note that $\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$. We therefore have

$$\int_{|z|=2} \frac{dz}{z^2-1} = \frac{1}{2}\left(\int_{|z|=2} \frac{dz}{z-1} - \int_{|z|=2} \frac{dz}{z+1}\right),$$

   and since the contour travels about each of these poles, we know that each of these terms equals $2\pi i$.

   (c) $\int_\Gamma \mathrm{Re}\,(z)\,dz$ where Γ is the line segment connecting $5i-7$ to $5i+9$.
   Do so both directly and by representing $\mathrm{Re}\,(z) = p(z)$ for some polynomial p, along Γ.
   Solution:
   We can directly define $\gamma(t) = 5i + t$ for $t \in [-7, 9]$, and compute

$$\int_\gamma \mathrm{Re}\,(z)\,dz = \int_{-7}^9 tdt = \left.\frac{t^2}{2}\right|_{-7}^9 = 16.$$

   Alternatively, along Γ we note that $\mathrm{Re}\,(z) = z - 5i$, and therefore

$$\int_\Gamma \mathrm{Re}\,(z)\,dz = \int_\Gamma (z-5i)dz = \left.\left(\frac{z^2}{2} - 5iz\right)\right|_{5i-7}^{5i+9} = \left.\frac{z}{2}(z-10i)\right|_{5i-7}^{5i+9} =$$
$$= \frac{1}{2}\left(|5i+9|^2 - |5i-7|^2\right) = 16$$

(d) $\int_\Gamma \mathrm{Re}\,(z)\,dz$ where Γ is the circle of radius $r > 0$ about 0.
Do so both directly and by representing $\mathrm{Re}\,(z) = \frac{p(z)}{q(z)}$ for some two polynomials p, q, along Γ.
Solution:
We can define $\gamma(t) = re^{it}$ for $t \in [0, 2\pi]$ and compute

$$\int_\gamma \mathrm{Re}\,(z)\,dz = \int_0^{2\pi} r\cos(t) ir\left(\cos(t) + i\sin(t)\right) dt =$$
$$= ir^2 \int_0^{2\pi} \frac{1+\cos(2t)}{2} dt - r^2 \int_0^{2\pi} \frac{\sin(2t)}{2} dt = \pi i r^2$$

Alternatively, we note that along Γ we have $\mathrm{Re}\,(z) = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(z + \frac{r^2}{z})$, so that

$$\int_\Gamma \mathrm{Re}\,(z)\,dz = \frac{1}{2}\int_\Gamma z dz + \frac{r^2}{2}\int_\Gamma \frac{1}{z} dz = 0 + \frac{r^2}{2} 2\pi i = \pi i r^2$$

2. Evaluate the line integral $\int_\gamma \frac{dz}{z}$ where

$$\gamma(t) = \begin{cases} -1 + i + e^{-it} & 0 \le t < \frac{\pi}{2} \\ -1 - i + e^{i(\pi - t)} & \frac{\pi}{2} \le t \le \pi \end{cases}$$

Solution:
We observe that $\frac{1}{z}$ is a holomorphic function in a neighborhood of the domain bounded by γ and $\alpha(t) = e^{it}$ for $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. We can therefore use Cauchy's theorem to integrate along α instead:

$$\int_\alpha \frac{dz}{z} = \int_{\pi/2}^{3\pi/2} \frac{ie^{it}}{e^{it}} dt = \pi i.$$

Alternatively, those who've studied holomorphic branches of the logarithm function will recall that one exists in $\mathbb{C} \setminus [0, \infty)$, given by $\mathrm{Ln}(re^{i\theta}) = \ln(r) + i\theta$, where θ is taken to be between 0 and 2π. Its derivative in that domain is $\frac{1}{z}$, and by the fundamental theorem of calculus we have

$$\int_\gamma \frac{dz}{z} = \mathrm{Ln}(\gamma(\pi)) - \mathrm{Ln}(\gamma(0)) = i\frac{3\pi}{2} - i\frac{\pi}{2} = \pi i.$$

3. Evaluate the line integral $\int_\gamma \left(e^{1/\bar{z}} + \frac{1}{e^{1/\bar{z}}}\right) dz$ where $\gamma(t) = re^{it}$ for some $r > 0$, and $t \in [0, \pi]$.
Solution:
We note that along γ one has $\frac{1}{\bar{z}} = \frac{z}{r^2}$, and so $e^{1/\bar{z}} + \frac{1}{e^{1/\bar{z}}} = e^{z/r^2} + e^{-z/r^2} = 2\cosh(\frac{z}{r^2})$. That last term is an entire function, with a primitive $2r^2 \sinh \frac{z}{r^2}$, and so we have:

$$\int_\gamma \left(e^{1/\bar{z}} + \frac{1}{e^{1/\bar{z}}}\right) dz = 2r^2\left(\sinh(\gamma(\pi)) - \sinh(\gamma(0))\right) = 2r^2(\sinh \frac{-1}{r} - \sinh \frac{1}{r}) = -4r^2 \sinh \frac{1}{r}.$$

4. (a) Suppose $\Omega \subseteq \mathbb{C}$ has a boundary consisting of a finite number of simple, closed, piecewise C^1 curves. Assume that $f(x+iy) = u(x,y) + iv(x,y)$ is defined in a neighborhood of $\overline{\Omega}$, and that u, v have continuous partial derivatives in that neighborhood. Using Green's theorem, express $\int_{\partial\Omega} f(z)dz$ in terms of the partial derivatives of u, v.
Caution: f needn't be holomorphic.
Solution:
Without loss of generality, we assume $\partial\Omega$ consists of a single closed curve, $\gamma : [a,b] \to \mathbb{C}$. The extension to the general case is a simple matter of summing line integrals over each curve.
Using what we know of line integrals in $\mathbb{R}^2$ and Green's theorem, we denote $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ where $\gamma_1, \gamma_2 : [a,b] \to \mathbb{R}$, and note that:

$$\int_{\partial\Omega} f(z)dz = \int_a^b (u(\gamma(t)) + iv(\gamma(t)))(\gamma_1'(t) + i\gamma_2'(t))\,dt =$$
$$= \int_a^b (u(\gamma(t))\gamma_1'(t) - v(\gamma(t))\gamma_2'(t))\,dt + i\int_a^b (u(\gamma(t))\gamma_2'(t) + v(\gamma(t))\gamma_1'(t))\,dt =$$
$$= \int_\gamma udx - vdy + i\int_\gamma vdx + udy = -\int_\Omega \left(v_x' + u_y'\right)dxdy + i\int_\Omega \left(u_x' - v_y'\right)dxdy$$

(b) Show that the area of every triangle $\Delta \subseteq \mathbb{C}$ is given by $S(\Delta) = \frac{1}{2i}\int_{\partial\Delta} \bar{z}dz$.
Solution:
If $f(z) = \bar{z}$ then $u(x,y) = x$ and $v(x,y) = -y$. In particular, by the last part

$$2iS(\Delta) = i\int_\Delta 2dxdy = -\int_\Delta \left(v_x' + u_y'\right)dxdy + i\int_\Delta \left(u_x' - v_y'\right)dxdy = \int_{\partial\Delta} \bar{z}dz.$$

5. By integrating $\frac{R+z}{z(R-z)}$ along a suitable circle, show that

$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}d\theta = 2\pi,$$

for some $0 < r < R$. This will fact will come in handy in various situations.
Solution:
We start by noting that $\frac{R+z}{z(R-z)} = \frac{1}{z} + \frac{2}{R-z}$, therefore if we let $\gamma(t) = re^{it}$ for $t \in [0, 2\pi]$ we have

$$\int_\gamma \frac{R+z}{z(R-z)}dz = \int_\gamma \frac{dz}{z} - 2\int_\gamma \frac{dz}{z-R} = 2\pi i + 0.$$

On the other hand, we can directly compute

$$\int_\gamma \frac{R+z}{z(R-z)}dz = \int_0^{2\pi} \frac{R+re^{it}}{re^{it}(R-re^{it})}ire^{it}dt = i\int_0^{2\pi} \frac{(R+re^{it})(R-re^{-it})}{(R-re^{it})(R-re^{-it})}dt =$$
$$= i\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}dt - \int_0^{2\pi} \frac{2Rr\sin t}{R^2 - 2Rr\cos t + r^2}dt$$

It's left to verify that the real part of that last integral vanishes. We can do so either by remembering that we already know it's purely imaginary, or by identifying the integrand as an odd function around $t = \pi$

6. As you'll recall, Weierstrass's theorem states that every continuous function $f : [a,b] \to \mathbb{R}$ can be uniformly approximated by polynomials. That is, there exist polynomials $p_n(x)$ such that $\sup_{x\in[a,b]} |f(x) - p_n(x)| \underset{n\to\infty}{\longrightarrow} 0$.

   Use integration along curves to prove that the same does not hold for $\frac{1}{z}$ when viewed as a function on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (which is, just as $[a,b]$ is for $\mathbb{R}$, a compact subset of the complex plain). More explicitly, find $\varepsilon > 0$ such that for every polynomial $p(z)$ it holds that $\sup_{z\in S^1} \left|\frac{1}{z} - p(z)\right| \geq \varepsilon$.

   Hint: consider closed contours about the origin.

   Solution:

   Suppose we have a polynomial $p(z)$ such that $M = \sup_{z\in S^1} \left|\frac{1}{z} - p(z)\right| < 1$. Then we may note that

$$2\pi = |2\pi i - 0| = \left|\int_{S^1} \left(\frac{1}{z} - p(z)\right) dz\right| = \left|\int_0^{2\pi} \left(\frac{1}{e^{it}} - p(e^{it})\right) ie^{it} dt\right| \leq$$
$$\leq \int_0^{2\pi} \left|\frac{1}{e^{it}} - p(e^{it})\right| dt \leq \int_0^{2\pi} M dt < 2\pi$$

   It follows that for every polynomial $p(z)$ one has $\sup_{z\in S^1} \left|\frac{1}{z} - p(z)\right| \geq 1$.

7. In each of the following cases, explain why the given line integral vanishes:

   (a) $\int_\gamma \frac{dz}{(z-3)^3}$ where $\gamma(t) = i + 4e^{it}$ for $t \in [0, 2\pi]$.

   Solution:

   $\frac{1}{(z-3)^3}$ has the primitive $\frac{-1}{2(z-3)^2}$ in $\mathbb{C} \setminus \{3\}$, hence every line integral along a closed curve vanishes.

   (b) $\int_\gamma z\,|z|\,dz$ where $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

   Solution:

   Along γ it holds that $z\,|z| = z$, which is an entire function.

   (c) $\int_\gamma \frac{e^{z^2}}{z^2+4} dz$ where $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

   Solution:

   $\frac{e^{z^2}}{z^2+4} = \frac{e^{z^2}}{(z-2i)(z+2i)}$ is holomorphic in $D_{3/2}(0)$, for example, which contains the image of γ along with its interior.

   (d) $\int_\gamma \frac{1}{\sin^2 z} dz$ where $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

   Solution:

   The function $\cot z = \frac{\cos z}{\sin z}$ is holomorphic on $\mathbb{C} \setminus 2\pi\mathbb{Z} = \{z \in \mathbb{C} \mid \nexists k \in \mathbb{Z} : z = 2\pi k\}$ (which is to say, $\sin z$ only vanishes on integer multiples of 2π). Its derivative there is $\frac{-1}{\sin^2 z}$, and therefore line integrals along closed curves on $\frac{1}{\sin^2 z}$ vanish.

8. Let $\Omega \subset \mathbb{C}$ be a region, and $f : \Omega \to \mathbb{C}$ a holomorphic function. Suppose $\gamma : [a,b] \to \Omega$ is a closed simple contour. We'll show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary.

   (a) Use the Cauchy-Riemann equations to express the real and imaginary parts of the integrand $\overline{f(z)} f'(z)$.

   (b) Utilize your expression to show that

$$\mathrm{Re}\left(\int_\gamma \overline{f(z)} f'(z) dz\right) = \int_a^b \left[\left(u(\gamma(t))u'_x(\gamma(t)) + v(\gamma(t))v'_x(\gamma(t))\right) \mathrm{Re}\left(\gamma'(t)\right) + \right.$$
$$\left. + \left(u(\gamma(t))u'_y(\gamma(t)) + v(\gamma(t))v'_y(\gamma(t))\right) \mathrm{Im}\left(\gamma'(t)\right)\right] dt$$

(c) Group the elements of the integrand and utilize C-R again to express it as

$$(u \circ \gamma)(t)(u \circ \gamma)'(t) + (v \circ \gamma)(t)(v \circ \gamma)'(t).$$

(d) Deduce the claim.

Solution:
If $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic then we know that

$$f'(x+iy) = u'_x(x,y) + iv'_x(x,y) = v'_y(x,y) - iu'_y(x,y),$$

and we therefore note

$$\begin{aligned}\int_\gamma \overline{f(z)} f'(z)dz &= \int_\gamma (u(z) - iv(z)) (u'_x(z) + iv'_x(z))\, dz = \\ &= \int_\gamma \left[(u(z)u'_x(z) + v(z)v'_x(z)) - i\left(u(z)u'_y(z) + v(z)v'_y(z)\right)\right] dz = \\ &= \int_a^b [(u(\gamma(t))u'_x(\gamma(t)) + v(\gamma(t))v'_x(\gamma(t)))\operatorname{Re}(\gamma'(t)) + \\ &\qquad + \left(u(\gamma(t))u'_y(\gamma(t)) + v(\gamma(t))v'_y(\gamma(t))\right)\operatorname{Im}(\gamma'(t))\big] + i[\ldots]\, dt\end{aligned}$$

Now, the imaginary part of this integral doesn't concern us; Our aim is to show that the real part vanishes. Let's take another look at the integrand, then, and note that if $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ then $\operatorname{Re}(\gamma'(t)) = \gamma_1'(t)$ and $\operatorname{Im}(\gamma'(t)) = \gamma_2'(t)$:

$$\begin{aligned}&(u(\gamma(t))u'_x(\gamma(t)) + v(\gamma(t))v'_x(\gamma(t)))\operatorname{Re}(\gamma'(t)) + \\ &\qquad\qquad + \left(u(\gamma(t))u'_y(\gamma(t)) + v(\gamma(t))v'_y(\gamma(t))\right)\operatorname{Im}(\gamma'(t)) = \\ &= (u\circ\gamma)(t)\left(u'_x(\gamma_1(t),\gamma_2(t))\gamma_1'(t) + u'_y(\gamma_1(t),\gamma_2(t))\gamma_2'(t)\right) + \\ &\qquad\qquad + (v\circ\gamma)(t)\left(v'_x(\gamma_1(t),\gamma_2(t))\gamma_1'(t) + v'_y(\gamma_1(t),\gamma_2(t))\gamma_2'(t)\right) = \\ &\qquad = (u\circ\gamma)(t)(u\circ\gamma)'(t) + (v\circ\gamma)(t)(v\circ\gamma)'(t),\end{aligned}$$

where the last stage utilizes the chain rule. Therefore we can now say that

$$\begin{aligned}\operatorname{Re}\left(\int_\gamma \overline{f(z)}f'(z)dz\right) &= \int_a^b [(u\circ\gamma)(t)(u\circ\gamma)'(t) + (v\circ\gamma)(t)(v\circ\gamma)'(t)]\, dt = \\ &= (u\circ\gamma)(t)\big|_a^b + (v\circ\gamma)(t)\big|_a^b = 0 + 0,\end{aligned}$$

since γ is a closed curve.

9. In this nonobligatory exercise we prove some of the properties of the *winding number*. Let $\gamma : [a,b] \to \mathbb{C}$ is a closed contour, and suppose $w \in \mathbb{C}$ isn't in the image of γ. We define the *winding number* of γ with respect to w, or the *index* of w with respect to γ, as $n(w,\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$.

(a) Show that $n(\cdot,\gamma) : (\mathbb{C}\setminus\gamma[a,b]) \to C$ is a continuous function.
Solution:
Take $w \in \mathbb{C}\setminus\gamma[a,b]$, and define $R = \min\{|w-\gamma(t)| \mid t\in[a,b]\}$ (why does a minimum exist, and why isn't it zero?). Then if $\delta < \frac{R}{2}$ then for all $w' \in D_\delta(w)$ and for all $t \in [a,b]$ it holds that

$$|\gamma(t) - w'| \geq |\gamma(t) - z| - |z - w'| \geq R - \frac{R}{2} = \frac{R}{2},$$

and therefore $\min\{|w' - \gamma(t)| \mid t \in [a,b]\} \geq \frac{R}{2}$. Now, we can simply estimate:

$$\begin{aligned}|n(w,\gamma) - n(w',\gamma)| &= \frac{1}{2\pi}\left|\int_a^b \left(\frac{1}{\gamma(t)-w} - \frac{1}{\gamma(t)-w'}\right)\gamma'(t)dt\right| = \\ &= \frac{1}{2\pi}\left|\int_a^b \frac{w-w'}{(\gamma(t)-w)(\gamma(t)-w')}\gamma'(t)dt\right| \leq \\ &\leq \frac{|w-w'|}{2\pi}\int_a^b \frac{|\gamma'(t)|}{|\gamma(t)-w|\,|\gamma(t)-w'|}dt \leq \\ &\leq \frac{|w-w'|}{\pi R^2}\int_a^b |\gamma'(t)|\,dt = |w-w'|\,\frac{L(\gamma)}{\pi R^2}\end{aligned}$$

It will therefore suffice to choose $\delta = \min\left\{\frac{R}{2}, \frac{\pi R^2 \varepsilon}{L(\gamma)}\right\}$, and for all $w' \in D_\delta(w)$ we've seen that $|n(w,\gamma) - n(w',\gamma)| \leq \varepsilon$.

(b) Show that $n(w,\gamma)$ is always an integer. That is, that $n(\cdot,\gamma) : (\mathbb{C} \setminus \gamma[a,b]) \to \mathbb{Z}$. Hint: one way of achieving this feat is by considering the function $h(t) = \int_a^t \frac{\gamma'(\tau)}{\gamma(\tau)-w}d\tau$. Verify that it is a piecewise-differentiable continuous function, and then examine the derivative of $e^{-h(t)}(\gamma(t)-w)$. Deduce the claim from from what you discover.

<u>**Solution**</u>:

We define $h : [a,b] \to \mathbb{C}$ by $h(t) = \int_a^t \frac{\gamma'(\tau)}{\gamma(\tau)-w}d\tau$. Note that the integrand is a bounded piecewise-continuous function (since γ is piecewise C^1), and therefore h is well-defined and continuous. We also know from the fundamental theorem (of real calculus) that $h'(t) = \frac{\gamma'(t)}{\gamma(t)-w}$ wherever it's continuous (i.e., wherever γ is differentiable). In conclusion, h is a piecewise C^1 continuous mapping, and therefore so is $g(t) = e^{-h(t)}(\gamma(t)-w)$. Wherever h has a derivative we know that

$$g'(t) = -h'(t)e^{-h(t)}(\gamma(t)-w)+e^{-h(t)}\gamma'(t) = e^{-h(t)}\left(\gamma'(t) - \frac{\gamma'(t)}{\gamma(t)-w}(\gamma(t)-w)\right) = 0$$

It follows that g is constant, and therefore for all $t \in [a,b]$ it holds that

$$e^{-h(t)}(\gamma(t)-w) = g(t) = g(a) = e^{-h(a)}(\gamma(a)-w) = \gamma(a)-w.$$

Put differently, $e^{h(t)} = \frac{\gamma(t)-w}{\gamma(a)-w}$. Since γ is closed, we have $e^{h(b)} = 1$, hence $h(b)$ is an integer multiple of $2\pi i$.

(c) Show that if w_1, w_2 are connected by a path α whose image is disjoint from γ's, then

$$n(w_1,\gamma) = n(w_2,\gamma).$$

<u>**Solution**</u>:

Let $k = n(w_1,\gamma)$,$m = n(w_2,\gamma)$, and without loss of generality assume $k \leq m$. We shall assume that $k < m$. Suppose that $\alpha : [c,d] \to \mathbb{C}$ is a path such that $\alpha(c) = w_1$,$\alpha(d) = w_2$ and $\alpha[c,d] \cap \gamma[a,b] = \emptyset$. Then we can define $\beta : [c,d] \to \mathbb{Z}$ by $\beta(t) = n(\alpha(t),\gamma)$, and as a composition of continuous function we know that β is continuous, $\beta(c) = k$ and $\beta(d) = m$. So there exists $t_0 \in (c,d)$ such that $\beta(t_0) = k+1$.

Let $s = \inf\{t \in (c,t_0] \mid \beta(t) = k+1\}$. We know that $c \leq s \leq t_0$. Note that if $s < t_0$ then by necessity $\beta(s) = \lim_{t \searrow s}\beta(t) = \lim_{t \searrow s}(k+1) = k+1$ (and if $s = t_0$ then $\beta(s) = k+1$). Similarly, if $s > c$ then $\beta(s) = \lim_{t \nearrow s}\beta(t) = \lim_{t \nearrow s} k = k$ (and if $s = c$ then $\beta(s) = k$). This in contradiction, therefore $k = m$.

10. In class, we've proved a variant of the Cauchy-Goursat theorem which assumes the continuity of $f'(z)$ for some holomorphic function f. In fact, the assumption that f' is continuous is redundant (as we'll soon see, it's actually a consequence of f being holomorphic). In this exercise, we shall prove that for integration over rectangles. As in problem 9, this exercise is non-mandatory.

    (a) Let $R \subseteq \mathbb{C}$ be a rectangle. Explain why $\int_{\partial R} dz = \int_{\partial R} z dz = 0$.

    (b) Let $R \subseteq \mathbb{C}$ be a rectangle, and f a holomorphic function on a neighborhood of R. Divide R into 4 identical rectangles $R^{(j)}$ for $j = 1, \ldots, 4$. Show that there exists j such that $\left|\int_{\partial R^{(j)}} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)dz\right|$.

    (c) Use that fact to arrive by a sequence of rectangles $R = R_0, R_1, \ldots, R_n, \ldots$, each taking a quarter of the area of the preceding one, such that

    $$\left|\int_{\partial R_n} f(z)dz\right| \geq \frac{1}{4^n}\left|\int_{\partial R} f(z)dz\right|.$$

    Explain the existence of (a single) $z_0 \in \mathbb{C}$ such that $z_0 \in R_n$ for all n.

    (d) Find some $M > 0$ such that for every $\varepsilon > 0$ you can use the existence of $f'(z_0)$, as well as the first part of this question, to come by the bound $\left|\int_{\partial R_n} f(z)dz\right| \leq \varepsilon \frac{M}{4^n}$ (for a large enough n).
    Note: M must not depend on n.

    (e) Deduce the claim.

Solution:
We begin by noting that $(z)' = 1$ and $(\frac{1}{2}z^2)' = z$ in the entire plain, hence integrals along closed curves on the integrands 1,z vanish. In particular, for every rectangle we have $\int_{\partial R} dz = \int_{\partial R} z dz = 0$.

Now, for a general holomorphic function f on a neighborhood of a rectangle R, if one divides R into four identical rectangles $R^{(j)}$, $j = 0, \ldots, 3$, then it is a simple matter of applying definitions to verify that $\int_{\partial R} f(z)dz = \sum_{j=0}^{3} \int_{\partial R^{(j)}} f(z)dz$. Hence one observes

$$\left|\int_{\partial R} f(z)dz\right| \leq \sum_{j=0}^{3}\left|\int_{\partial R^{(j)}} f(z)dz\right|,$$

implying that for at least one j $\left|\int_{\partial R^{(j)}} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)dz\right|$.

We can therefore denote $R_0 = R$, and define, inductively for all $n \in \mathbb{N}$, $R_{n+1} = R_n^{(j)}$ for such j as $\left|\int_{\partial R_{n+1}} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R_n} f(z)dz\right|$. It readily follows that for all $n \in \mathbb{N}$

$$\left|\int_{\partial R_n} f(z)dz\right| \geq \frac{1}{4^n}\left|\int_{\partial R_0} f(z)dz\right|.$$

This is a sequence of rectangles where each is contained in its predecessor and its edges are half the length of its predecessor's. Therefore the vertices of those rectangles form a Cauchy sequence, which converges to a single $z_0 \in \mathbb{C}$. Note that $z_0 \in R_n$ for all $n \in \mathbb{N}$, and that it's the unique such point (why?).

Next, since $f'(z_0)$ exists (f is holomorphic in a neighborhood of R_0), for every $\varepsilon > 0$ we can find $\delta > 0$ such that when $|z - z_0| < \delta$ one has $\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$. Put differently, this last inequality means that $|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$.

Now we find $n \in \mathbb{N}$ such that $R_n \subseteq D_\delta(z_0)$ (why does one exist?) and estimate

$$\begin{aligned}
\left|\int_{\partial R_n} f(z)dz\right| &= \left|\int_{\partial R_n} f(z)dz - (f(z_0) + f'(z_0)z_0)\int_{\partial R_n} dz - f'(z_0)\int_{\partial R_n} zdz\right| \\
&= \left|\int_{\partial R_n} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right] dz\right| \\
&\leq \sup_{z\in\partial R_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \cdot \mathrm{L}(\partial R_n) \\
&\leq \varepsilon \sup_{z\in\partial R_n} |z - z_0| \cdot \mathrm{L}(\partial R_n) \leq \varepsilon d_n \,\mathrm{L}(\partial R_n),
\end{aligned}$$

where d_n is the length of diagonal of R_n, and $\mathrm{L}(\partial R_n)$ the perimeter of R_n.
It remains to note that $d_n = \frac{1}{2^n} d_0$ and $\mathrm{L}(\partial R_n) = \frac{1}{2^n}\,\mathrm{L}(\partial R_n)$, hence

$$\frac{1}{4^n}\left|\int_{\partial R_0} f(z)dz\right| \leq \left|\int_{\partial R_n} f(z)dz\right| \leq \varepsilon d_n \,\mathrm{L}(\partial R_n) = \frac{1}{4^n}\varepsilon d_0 \,\mathrm{L}(\partial R_0),$$

and since this holds for all $\varepsilon > 0$, $\int_{\partial R_0} f(z)dz = 0$.